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Problems and Solutions

PROBLEMS

Proposals

To be considered for publication, solutions should be received by July 1, 2020.

2086. *Proposed by David M. Bradley, University of Maine, Orono, ME.*

Let $f(k)$ denote the largest integer that is a divisor of $n^k - n$ for all integers n. For example, $f(2) = 2$ and $f(3) = 6$. Determine $f(k)$ for all integers $k > 1$.

2087. *Proposed by Florin Stanescu, Şerban Cioiculescu School, Găeşti, Romania.*

Consider the sequence defined by $x_1 = a > 0$ and

$$x_n = \ln\left(1 + \frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right) \quad \text{for} \quad n \geq 2.$$

Compute $\lim_{n\to\infty} x_n \ln n$.

2088. *Proposed by Mircea Merca, University of Craiova, Romania.*

Let n and t be nonnegative integers. Prove that

$$\sum_{k=0}^{2n} (-1)^k F_{tk} F_{2tn-tk} = -\frac{F_t}{L_t} F_{2tn},$$

where F_i denotes the ith Fibonacci number and L_i denotes the ith Lucas number.

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We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this Magazine.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at `http://mathematicsmagazine.submittable.com`. *More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by LaTeX source. General inquiries to the editors should be sent to* `mathmagproblems@maa.org`.

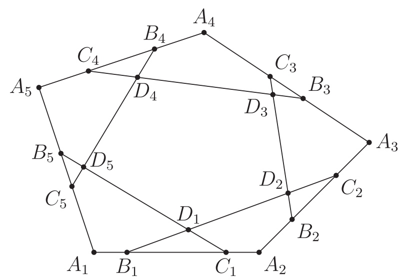
2089. *Proposed by Rick Mabry, LSU Shreveport, Shreveport, LA.*

Let $A_1, A_2, \ldots, A_n$ be the vertices of a convex n-gon in the plane. Identifying the indices modulo n, define the following points: Let B_i and C_i be vertices on $\overline{A_i A_{i+1}}$ such that

$$A_i B_i = C_i A_{i+1} < \frac{A_i A_{i+1}}{2},$$

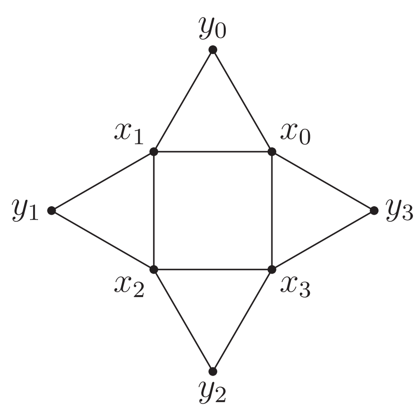
and let D_i be the intersection of $\overline{B_{i-1} C_i}$ and $\overline{B_i C_{i+1}}$. Prove that

$$\prod_{i=1}^{n} \frac{B_i D_i}{D_i C_i} = 1.$$



2090. *Proposed by Gregory Dresden, Washington & Lee University, Lexington, VA.*

Recall that a *matching* of a graph is a set of edges that do not share any vertices. For example, C_4, the cyclic graph on four vertices (i.e., a square), has seven matchings: the empty set, single edges (four of these), or pairs of opposite edges (two of these).
Let G_n be the (undirected) graph with vertices x_i and y_i, $0 \le i \le n-1$, and edges $\{x_i, x_{i+1}\}$, $\{x_i, y_i\}$, and $\{y_i, x_{i+1}\}$, $0 \le i \le n-1$, where the indices are to be taken modulo n. For example, G_4 is shown below. Determine the number of matchings of G_n.



Quickies

1097. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let $z_1, \ldots, z_n \in \mathbb{C}$ with $|z_i| = 1$. Show that there exists $\omega \in \mathbb{C}$ with $|\omega| = 1$ such that $|(\omega - z_1)\ldots(\omega - z_n)| \geq 2$, and this result is the best possible, namely 2 cannot be replaced by any larger number.

1098. *Proposed by Oniciu Gheroghe, Botoşani, Romania.*

In the convex quadrilateral $ABCD$, $\angle BAD \cong \angle BCD$ both with measure $60°$. The diagonal $\overline{AC}$ bisects $\angle BAD$. Prove that $m(\angle BDA) = 2m(\angle BCA)$.

Solutions

2061. *Proposed by Florin Stanescu, Şerban Cioiculescu School, Găeşti, Romania.*

Three complex numbers a, b, c satisfy

$$|a| = |b| = |c| = 1 \qquad \text{and} \qquad a^3 + b^3 + c^3 = 2abc.$$

Prove that a, b, c are vertices of an isosceles triangle on the complex plane.

Solution by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela.
We shall prove more generally that, if

$$|a| = |b| = |c| = 1 \quad \text{and} \quad a^3 + b^3 + c^3 = rabc$$

for any real number r, $-1 \leq r \leq 3$, then a, b, c are the vertices of an isosceles triangle in the complex plane. Note that we are allowing for the possibility of degenerate triangles where vertices coincide.

A triangle is isosceles if and only if two of its central angles are congruent. There are three possibilities:

$$\frac{a}{b} = \frac{b}{c}, \quad \frac{b}{c} = \frac{c}{a}, \quad \text{or} \quad \frac{c}{a} = \frac{a}{b}.$$

Therefore the triangle is isosceles if and only if

$$(a^2 - bc)(b^2 - ca)(c^2 - ab) = 0, \text{ i.e.,}$$
$$a^4bc + ab^4c + abc^4 = a^3b^3 + b^3c^3 + a^3c^3.$$

If $a^3 + b^3 + c^3 = rabc$ then

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = \overline{a^3 + b^3 + c^3} = \overline{rabc} = \frac{r}{abc}.$$

Multiplying both sides by $(abc)^3$ we obtain

$$a^3b^3 + b^3c^3 + a^3c^3 = r(abc)^2 = a^4bc + ab^4c + abc^4$$

as desired.

Now suppose

$$a^3 + b^3 + c^3 = rabc$$

and, for example, $c/a = a/b$. Then

$$1 + \left(\frac{b}{a}\right)^3 + \left(\frac{c}{a}\right)^3 = \frac{rbc}{a^2} = r.$$

Since b/a and c/a are conjugate, we may put

$$\frac{b}{a} = e^{\phi i}, \quad \text{and} \quad \frac{c}{a} = e^{-\phi i}.$$

Then

$$e^{3\phi i} + e^{-3\phi i} = r - 1, \text{ i.e.,}$$

$$\cos 3\phi = \frac{r-1}{2}.$$

This shows that $-1 \le r \le 3$. Note that degenerate triangles with

$$\phi = 0, \text{(e.g., } a = b = c = 1) \quad \text{and} \quad \phi = \pi, \text{(e.g., } a = 1, b = c = -1)$$

can occur when $r = -1$ or 3.

In the original problem, $\cos 3\phi = 1/2$, so

$$\phi = \frac{\pi}{9}, \quad \frac{5\pi}{9}, \quad \text{or} \quad \frac{7\pi}{9},$$

and the angles of the triangle formed by a, b, and c are

$$\left(\frac{\pi}{18}, \frac{\pi}{18}, \frac{\pi}{9}\right), \quad \left(\frac{5\pi}{18}, \frac{5\pi}{18}, \frac{4\pi}{9}\right), \quad \text{or} \quad \left(\frac{7\pi}{18}, \frac{7\pi}{18}, \frac{2\pi}{9}\right).$$

Also solved by Robert A. Agnew, Hafez Al-Assad (Syria), Michel Bataille (France), Cal Poly Pomona Problem Solving Group, Robert Calcaterra, Adam Cofmann, Bruce E. Davis, Robert L. Doucette, George Washington University Problems Group, Kyle Gatesman, Michael Goldenberg & Mark Kaplan, Eugene A. Herman, Walther Janous (Austria), Stephen Kaczkowski, Koopa Tak Lun Koo (Hong Kong), Omran Kouba (Syria), Kee-Wai Lau (Hong Kong), Hyomin Park (Korea), Theophilus Pedapolu, Michael Reid, Ivan Retamoso, Leonel Robert & Charlotte Ochanine, Randy K. Schwartz, Daniel Vacaru (Romania), Lawrence R. Weill and the proposer. There was one incomplete or incorrect solution.

2062. *Proposed by Enrique Treviño, Lake Forest College, Lake Forest, IL.*

For every positive integer n, let $f(n)$ denote the number of occurrences of the digit 2 in the sequence $1, 2, \ldots, n$ of integers written in base 10. (For instance, $f(25) = 9$ because the digit 2 appears once in 2, 12, 20, 21, 23, 24, 25 and twice in 22.)

(i) Find a positive integer n such that $f(n) = n$.

(ii) Are there infinitely many solutions to $f(n) = n$?

Solution by Cassandra DeBacco (student) and Mark Capsambelis, Riverview High School, Oakmont, PA.
Let $n \in \mathbb{N}$. For all integers between 0 and $10^n - 1$, the digit 2 appears in each of the n decimal places $1/10$ of the time, so

$$f(10^n) = n10^{n-1}.$$

In particular, if $n = 10$, then

$$f\left(10^{10}\right) = 10 \cdot 10^9 = 10^{10},$$

which answers (i).

For (ii), note that for all $n > 100$,

$$f(10^n) = n10^{n-1} > 10^{n+1}.$$

So if $n > 100$ and $k \in \mathbb{N}$ such that $10^n \leq k < 10^{n+1}$, then

$$f(k) \geq f(10^n) > 10^{n+1} > k$$

since f is non-decreasing. Therefore, there are no solutions of $f(n) = n$ for $n > 100$, so there are only a finite number of solutions.

Also solved by Mohammed Aassila (France), Ulrich Abel (Germany), Hafez Al-Assad (Syria), Armstrong Problem Solvers, Brian Beasley, Virginia Faulkner Bolton, David Stone and John Hawkins, Robert Calcaterra, Bill Cowieson, Robert L. Doucette, Kyle Gatesman, George Washington University Problems Group, Eugene A. Herman, Kelly Jahns, Andrea McCormack, José Heber Nieto (Venezuela), Northwestern University Math Problem Solving Group, Charlotte Ochanine, Moubinool Omarjee (France), Timothy Prescott, Michael Reid, Arnold Saunders, Randy K. Schwartz, Allen Schwenk, James Swenson, Mark Wildon, and the proposer. There were 3 incomplete or incorrect solutions.

2063. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Evaluate

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{n+k-1}}{(n+k)^2}.$$

Solution by Northwestern University Math Problem Solving Group, Evanston, IL.
The answer is $\ln 2$.

Let S_N be the following partial sum:

$$S_N = \sum_{n=0}^{N}\sum_{k=1}^{\infty}\frac{(-1)^{n+k-1}}{(n+k)^2}.$$

For each fixed N we have that S_N is absolutely convergent, so we can rearrange terms and rewrite the sum as $S_N = S'_N + S''_N$, where (writing $n + k = j$):

$$S'_N = \sum_{\substack{n+k\leq N\\ n\geq 0,\, k\geq 1}}\frac{(-1)^{n+k-1}}{(n+k)^2} = \sum_{j=1}^{N} j\,\frac{(-1)^{j-1}}{j^2} = \sum_{j=1}^{N}\frac{(-1)^{j-1}}{j},$$

$$S''_N = (N+1)\sum_{j=N+1}^{\infty}\frac{(-1)^{j-1}}{j^2}.$$

The sum of the alternating series S''_N can be bound by its first term, so we have

$$|S''_N| < \frac{1}{N+1} \to 0 \quad \text{as} \quad N \to \infty.$$

Hence

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \frac{(-1)^{n+k-1}}{(n+k)^2} = \lim_{N\to\infty} S'_N + \lim_{N\to\infty} S''_N = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} + 0 = \ln 2\,.$$

Also solved by Ulrich Abel (Germany), Farrukh Rakhimjanovich Ataev (Uzbekistan), Michel Bataille (France), Necdet Batir (Turkey), Khristo Boyadzhiev, Brian Bradie, Robert Calcaterra, Hongwei Chen, Bill Cowieson, Bruce Davis, Robert L. Doucette, Saumya Dubey, John N. Fitch, Kyle Gatesman, Subhankar Gayen (India), George Washington University Problems Group, Tom Goebeler & Wendy Sun, Russelle Guadalupe (Philippines), GWstat Problem Solving Group, Eugene A. Herman, Walther Janous (Austria), Kee-Wai Lau (China), Pedro Acosta De Leon, Carl Libis, José Heber Nieto (Venezuela), Moubinool Omarjee (France), Emily Owen, Hyomin Park (Korea), Sumanth Ravipati, Edward Schmeichel, Randy K. Schwartz, Albert Stadler (Switzerland), Robert W. Vallin, Michael Vowe, Mark Wildon (UK), Lienhard Wimmer (Switzerland), John Zacharias, Yijie Zhu (China), and the proposer. There were 3 incomplete or incorrect solutions.

2064. *Proposed by Ioan Băetu, Botoşani, Romania.*

Characterize those integers $n \geq 2$ such that the ring $\mathbb{Z}_n$ of integers modulo n has a subset F that is a field under the operations of addition and multiplication induced from $\mathbb{Z}_n$. [Note that the unity i of such a field F need not be the unity 1 of $\mathbb{Z}_n$.]

Solution by Michael Reid, University of Central Florida, Orlando, FL.

The ring $\mathbb{Z}_n$ contains a subring F that is a field if and only if n is not powerful, i.e., there is a prime p such that n is divisible by p but not by p^2.

First suppose that p is a prime number such that $p|n$ but $p^2 \nmid n$. Write $n = pm$, so $p \nmid m$. By the Chinese remainder theorem, $\mathbb{Z}_n \cong \mathbb{Z}_p \times \mathbb{Z}_m$, and the subset $\mathbb{Z}_p \times \{0\}$ is isomorphic to $\mathbb{Z}_p$, which is a field.

Conversely, suppose $\mathbb{Z}_n$ contains a field F. Then F is finite, so its characteristic is a prime number p. The additive group $(F, +)$ has exponent p, so it is contained in the p-torsion subgroup of $(\mathbb{Z}_n, +)$. Denote this p-torsion subgroup by B. Since $(\mathbb{Z}_n, +)$ is a cyclic group, every subgroup, in particular B, is also cyclic. Moreover, $|B|$ is either p or 1, depending on whether p divides n or not. Hence, $p|n$, so B has order p, and F coincides with B. Write $n = mp^e$, where $e \geq 1$ and $p \nmid m$. Then the p-torsion subgroup of $\mathbb{Z}_n$ is precisely

$$B = \left\{0, mp^{e-1}, 2mp^{e-1}, \ldots, (p-1)mp^{e-1}\right\}.$$

If $e \geq 2$, then every element of $F = B$ is nilpotent, because

$$\left(amp^{e-1}\right)^2 = \left(a^2mp^{e-2}\right) mp^e \equiv 0 \bmod n.$$

This is a contradiction, so we must have $e = 1$, and thus $p^2 \nmid n$.

Also solved by Paul Budney, Robert Calcaterra, Bill Cowieson, and the proposer. There was one incomplete or incorrect solution.

2065. *Proposed by Su Pernu Mero, Valenciana GTO, Mexico.*

Let $\mathcal{Q}$ be a cube centered at the origin of $\mathbb{R}^3$. Choose a unit vector (a, b, c) uniformly at random on the surface of the unit sphere $a^2 + b^2 + c^2 = 1$, and let Π be the plane

$ax + by + cz = 0$ through the origin and normal to (a, b, c). What is the probability that the intersection of Π with $\mathcal{Q}$ is a hexagon?

Solution by Bill Cowieson, Fullerton College, Fullerton, CA.
The probability is

$$1 - \frac{6 \arcsin(1/3)}{\pi} \approx 0.350959.$$

Call a unit vector (a, b, c) "good" if $\Pi \cap \mathcal{Q}$ is a hexagon and "bad" otherwise. A vector is good if and only if Π intersects all six sides of $\mathcal{Q}$, therefore the regions of good and bad unit vectors share a boundary that consists of those unit vectors for which Π contains a vertex of $\mathcal{Q}$, i.e., those which are orthogonal to the vector from the origin to some vertex. Without loss of generality, let $\mathcal{Q}$ have vertices

$$(1, 1, 1),\ (1, 1, -1),\ (1, -1, 1),\ \ldots,\ (-1, -1, -1),$$

so this boundary consists of those unit vectors (a, b, c) which satisfy either

$$a + b + c = 0, \quad a + b - c = 0, \quad a - b + c = 0, \quad \text{or} \quad -a + b + c = 0.$$

By symmetry, it suffices to find the probability for (a, b, c) chosen from the positive octant $a, b, c > 0$, where the good/bad boundary equations are $a = b + c, b = c + a$, and $c = a + b$. These partition the positive octant of the sphere into 4 spherical triangles: a central triangle of good vectors

$$H = \{(a, b, c) \in \mathbb{R}^3 : a^2 + b^2 + c^2 = 1, a < b + c, b < c + a, c < a + b\},$$

which has vertices

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right), \quad \text{and} \quad \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right),$$

and three congruent triangles of bad vectors. The desired probability is $\text{Area}(H)/(\pi/2)$

On the unit sphere, the area of a spherical triangle is the sum of the three vertex angles minus π. Let θ be the common angle at each vertex of H, so $\text{Area}(H) = 3\theta - \pi$. The angle at $(\sqrt{2}/2, \sqrt{2}/2, 0)$ is between the planes $x = y + z$ and $y = x + z$, which is that between the normal vectors $(-1, 1, 1)$ and $(-1, 1, -1)$, so $\cos\theta = 1/3$,

$$\text{Area}(H) = 3 \arccos\left(\frac{1}{3}\right) - \pi = \pi/2 - 3 \arcsin\left(\frac{1}{3}\right),$$

and the probability that a random plane through the center of a cube makes a hexagon is

$$\frac{\text{Area}(H)}{\pi/2} = 1 - \frac{6 \arcsin(1/3)}{\pi}.$$

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert L. Doucette, John N. Finch, George Washington University Problems Group, J.A. Grzesik, Kidefumi Katsuura & Edward Schmeichel, Peter McPolin (Northern Ireland), Charlotte Ochanine, Randy K. Schwartz, Yawen Zhang (student), and the proposer. There were 4 incomplete or incorrect solutions.)

Answers

Solutions to the Quickies from page 73.

A1097. Define $P(z) = (z - z_1)\dots(z - z_n)$. For any complex number ω, we have

$$\frac{1}{n}\sum_{k=1}^{n} P\left(\omega \cdot e^{2ik\pi/n}\right) = \omega^n + (-1)^n z_1 \dots z_n.$$

Choose $\omega = -(z_1 \dots z_n)^{1/n}$. Then, by the formula above,

$$2 = |\omega^n + (-1)^n z_1 \dots z_n| = \left|\frac{1}{n}\sum_{k=1}^{n} P\left(\omega \cdot e^{2ik\pi/n}\right)\right| \le \frac{1}{n}\sum_{k=1}^{n}\left|P\left(\omega \cdot e^{2ik\pi/n}\right)\right|.$$

Therefore there exists $k \in \{1, \dots, n\}$ for which

$$\left|P\left(\omega \cdot e^{2ik\pi/n}\right)\right| \ge 2.$$

Taking $z_j = e^{2ij\pi/n}$, we have

$$|P(\omega)| = |\omega^n - 1| \le |\omega|^n + 1 = 2$$

for all ω with $|\omega| = 1$, so 2 cannot be replaced by any larger number.

A1098. Denote the circles that circumscribe $\triangle BCD$ and $\triangle ABD$ by $\mathcal{C}_1$ with center O and $\mathcal{C}_2$ with center O', respectively. One version of the law of sines states that $a/\sin A = b/\sin B = c/\sin C = 2R$, where R is the radius of the circle circumscribing the triangle. Since $\angle BAD \cong \angle BCD$, we conclude that $\mathcal{C}_1$ and $\mathcal{C}_2$ are congruent. The smaller arcs between B and D on $\mathcal{C}_1$ and $\mathcal{C}_2$ both have measure

$$2m(\angle BAD) = 2m(\angle BCD) = 120^\circ.$$

Therefore $\triangle OBO'$ and $\triangle ODO'$ are equilateral, so O lies on $\mathcal{C}_2$ (and O' lies on $\mathcal{C}_1$). Now O is the midpoint of arc BOD since $\overline{AC}$ is the angle bisector of $\angle BAD$. Hence we have

$$m(\angle BDA) = m(\angle BOA) = 2m(\angle BCO) = 2m(\angle BCA).$$